University of Memphis
Dept Math Sciences
D. P. Dwiggins, PhD

Ordinary Differential Equations
MATH 4120/6120
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Homework Assignment # 1

Due Tuesday Jan 31

We'll be doing a lot of proofs this semester, establishing methods of verifying that solutions to the differential equations under study will have certain properties, such as stability or boundedness. Thus, unfortunately, I will need to use this first assignment to assess how well everyone understands how to construct proofs and how to apply their results. When we get to Chapter 3 we'll have more opportunity to use computational techniques to achieve results.

1. Continuity and Uniform Continuity.

Let J be an interval (i.e. a connected subset of $\mathbb{R}$ with nonempty interior), and let t_0 be some interior point of J. Then we say a scalar function $x : J \to \mathbb{R}$ is continuous at t_0 when, for every error bound $\varepsilon > 0,$ there is some tolerance level $\delta > 0$ such that $\left|x(t) - x(t_0)\right| < \varepsilon$ when $\left|t - t_0\right| < \delta.$ If J has a real endpoint, say $t = a,$ with $t_0 > a$ for every interior point of J, then we say x is continuous at a from the right by requiring $\left|x(t) - x(a)\right| < \varepsilon$ when t satisfies $a < t < a + \delta.$ In this way we define one-sided continuity at any endpoints of J, and we then say x is continuous on J if x is continuous at every $t_0 \in J.$ Finally, if the value of δ does not depend on t_0, i.e. if δ can be written purely in terms of ε, then we say x is uniformly continuous on J.

(a) Consider $x(t) = 5 - 2t.$ Prove x is continous at $t_0 = 2.$ What value of δ corresponds to an error bound of $\varepsilon = 0.01$? Explain why, in this case, x is uniformly continous on all of $\mathbb{R}$.

(b) Consider $x(t) = t^2.$ Prove x is uniformly continuous on $J = (0,1).$ Explain why x is not uniformly continuous on all of $\mathbb{R}$.

(c) Consider $x(t) = 1/t.$ Prove x is continuous on $J = (0,1).$ In particular, if $t_0 = 0.5,$ what value of δ corresponds to $\varepsilon = 0.01$? Finally, explain why x is not uniformly continuous on J. (You might consider the sequence $t_0 = 0.1, t_0 = 0.01, t_0 = 0.001, \ldots$)

Many of our proof techniques involve finding sequences of approximate solutions to the differential equations, and then arguing those sequences must converge to something. We need to know that what they converge to will actually give the solutions we are looking for, and to do this we need to know our solution spaces are topologically complete. August Cauchy, a nineteenth-century mathematician, studied sequences which converge at a certain rate, and we now say a space is complete if every Cauchy sequence converges to something within the space. This is the topic of the next exercise.

2. Cauchy Sequences, Complete Spaces.

Let X be a vector space, such as the function space $X = C(J)$ (the space of functions which are continuous on a given interval J), or Euclidean space, $X = \mathbb{R}^n$, or simply just $X = \mathbb{R}$. These are all examples of the more general concept of a metric space, which means there is some way to measure distances between objects in the space. For $X = \mathbb{R}$, we simply use absolute value, i.e. the distance along the real number line between x and y is given by $|x-y|$. For $X = \mathbb{R}^n$, we let $\|x\|$ denote the norm (magnitude) of the vector x, and then define the distance between vectors x and y as $\|x-y\|$. There are many ways to define this norm, and there are many ways to define a norm on a function space, but for now let's assume the "distance" between two continuous scalar functions $x(t)$, $y(t)$ in $C(J)$ is given by $\|x-y\| = \max_{t \in J} |x(t) - y(t)|$.

In order to define what it means for a metric space to be complete, we first start with the concept of a Cauchy sequence. A given sequence $\{x_n\} \subset X$ satisfies the Cauchy criterion when, for every $\varepsilon > 0$, there exists some $N > 0$ such that $\|x_n - x_m\| < \varepsilon$ when $n > m > N$. We then say X is a complete metric space if every Cauchy sequence $\{x_n\} \subset X$ is also a convergent sequence, with $x_n \to x$ and $x \in X$. Spaces which contain Cauchy sequences which fail to converge to a point inside the space may then be called incomplete spaces. For example, the set of rational numbers, $\mathbb{Q}$, contains Cauchy sequences which do not converge to rational numbers, and so $\mathbb{Q}$ is incomplete. But if we include all the irrational numbers which are the convergent limits of sequences of rational numbers, then we have the set of real numbers. In other words, the betst way to define $\mathbb{R}$ is that it is the analytic completion of $\mathbb{Q}$.

(a) Let $\{a_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, i.e. $a_n = \frac{1}{n}$ is the sequence of the reciprocals of the natural numbers. Use the above definition to prove $\{a_n\}$ is a Cauchy sequence. Note: to establish the value of N which will guarantee $|a_n - a_m| < \varepsilon$, you'll probably want to use the absolute value property, $|x-y| \le |x| + |y|$. And if you want another hint, consider that $0 < x < M$ implies $\frac{1}{x} > \frac{1}{M}$. Once you've finished this problem, show you can use the exact same proof to explain why $b_n = \frac{n-1}{n}$ also defines a Cauchy sequence.

(b) Consider the sequence of functions, $\{x_n\} \subset C([0,1])$, defined by

$$x_n(t) = \begin{cases} 0 & , \quad 0 \le t \le \frac{n-1}{n} \\ nt - (n-1) & , \quad \frac{n-1}{n} < t \le 1 \end{cases}$$

Explain why each x_n is continuous, with $x_n(1) = 1$ for every n, and also explain why $\lim_{n\to\infty} x_n(t) = 0$ for $0 < t < 1$. For example, if $t = 0.9$, find $N > 0$ such that $x_n(t) = 0$ for all $n > N$. Once you have done this, you will have shown that $\{x_n\}$ converges to the function $x(t)$ which is equal to 0 for $0 \le t < 1$, with $x(t) = 1$, and so $\{x_n\}$ converges to a function which does not belong to $C([0,1])$.

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(c) We would not be able to do very much if $C([0,1])$ turned out to not be a complete metric space. Explain why the example in part (b) does not mean $C([0,1])$ is incomplete. You might first consider the following computation: Let $t = 0.999$, and find a value of m such that $x_m(t) = 0$. Next, find a value of $n > m$ with $x_n(t) \geq 0.5$. You can use this example to justify the argument that no value of N is going to work for $\varepsilon < 0.5$.

3. Existence and Uniqueness of Solutions.

Consider the initial value problem $x' = x^{1/3}$, with initial condition $x_0 = 0$ at $t_0 = 0$. Show there are two different solutions to this problem. What property of this equation fails the uniqueness part of the Fundamental Theorem?

4. Continuation of Solutions.

Given the differential equation $x' = x^2$, explain why there is a unique solution for every initial condition (t_0, x_0). Find the solutions for each of the following initial conditions:

(a) $(t_0, x_0) = (0,1)$ (b) $(t_0, x_0) = (1,0)$ (c) $(t_0, x_0) = (1,1)$

You should be able to determine there is only one solution to this equation which is defined for all $t \in \mathbb{R}$, while every other solution has only a finite interval of existence $J = (t_0, t_0 + T)$, where the size of T depends on the initial condition. What do you think might be the reason why this particular differential equation automatically suggests solutions cannot be continued? Would you say this equation satisfies the condition that solutions depend continuously upon the initial conditions? Why or why not?

5. Fundamental Theorem for Second-Order Equations.

Given continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ and real numbers t_0, y_0, and y_0', consider the initial value problem $y'' + f(y)y' + g(y) = 0$, with initial conditions $y(t_0) = y_0$, $y'(t_0) = y_0'$. Prove this problem has a unique solution for every initial condition, as long as we also have $f \in C^1(\mathbb{R})$ and $g \in C^1(\mathbb{R})$.

The way to construct this proof is to write the IVP in the form $x' = F(x)$, $x(t_0) = x_0$, where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is defined by taking $x_1 = y, x_2 = y'$, and then using the fundamental theorem for first-order ODE's. The way to show $F(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$ is continuous is to explain why both f_1 and f_2 are continuous, and the way to show $\dfrac{\partial F}{\partial x}$ is continuous is to investigate the continuity of the four partial derivatives $\dfrac{\partial f_i}{\partial x_j}$, $i = 1,2$, $j = 1,2$.