University of Memphis
MATH 3242
Linear Algebra
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Dwiggins

Homework Assignment # 4

Chapter Seven:
Eigensystems for a Matrix

2. $A = \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$ has trace equal to 2 + (–2) = 0 and determinant (–4) – (–4) = 0,

and so this means that $\lambda = 0$ is the only eigenvalue for A. Also, A has only one eigenvector, namely $\mathbf{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, which serves as the basis vector for the null space of A. Note this vector is orthogonal to (2, 1), the basis for the row space.

4. $A = \begin{pmatrix} -4 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ is a triangular matrix, and so its eigenvalues are given by

the diagonal elements, i.e. $\lambda_1 = -4$, $\lambda_2 = 1$, and $\lambda_3 = 3$. Using $\mathbf{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$, the

equation $A\mathbf{x} = \lambda\mathbf{x}$ gives the system $-4u + v + 2w = \lambda u$, $v + w = \lambda v$, and $3w = \lambda w$.

Thus, we see w can have any value when $\lambda = 3$, but otherwise we must have $w = 0$.

When $\lambda = -4$, setting $w = 0$ in the second equation gives $v = -4v$, and so $v = 0$, and the first equation becomes $-4u = -4u$, which means we can just take $u = 1$.

When $\lambda = 1$, setting $w = 0$ in the second equation gives $v = v$, so v can have any value, and the first equation gives $-4u + v = u$, or $v = 5u$, so we can take $u = 1$ and $v = 5$.

Finally, when $\lambda = 3$ the first two equations can be written as $v + 2w = 7u$ and $w = 2v$, where w can have any value other than zero, since an eigenvector cannot be the zero vector. Substituting $w = 2v$ into the first equation gives $5v = 7\underline{u}$, and so the eigenvector using whole numbers is most easily written using $u = 5$ and $v = 7$, which then gives $w = 2v = 14$.

Thus, the eigenvectors may be given by $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 5 \\ 7 \\ 14 \end{pmatrix}$.

Check: $A\mathbf{x}_1 = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} = -4\mathbf{x}_1$, $A\mathbf{x}_2 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix} = \mathbf{x}_2$ (with $\lambda_2 = 1$),

and $A\mathbf{x}_3 = \begin{pmatrix} -20 + 7 + 28 \\ 0 + 7 + 14 \\ 0 + 0 + 42 \end{pmatrix} = \begin{pmatrix} 15 \\ 21 \\ 42 \end{pmatrix} = 3\mathbf{x}_3$, as required.

6. Given $A = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{pmatrix}$, in order to find the eigenvalues we compute the

determinant $|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & -4 \\ 0 & \lambda - 1 & 2 \\ -1 & 0 & \lambda + 2 \end{vmatrix} = (\lambda - 1)\cdot[(\lambda - 1)(\lambda + 2) - 0] + (-4)\cdot[0 + (\lambda - 1)]$

$= (\lambda - 1)\cdot[(\lambda - 1)(\lambda + 2) - 4] = (\lambda - 1)(\lambda^2 + \lambda - 6) = (\lambda - 1)(\lambda - 2)(\lambda + 3)$.

Setting this equal to zero gives the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = -3$.

Again taking $\mathbf{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ in the equation $A\mathbf{x} = \lambda\mathbf{x}$, we obtain the system

$\left\{ \begin{aligned} u + 4w &= \lambda u \\ v - 2w &= \lambda v \\ u - 2w &= \lambda w \end{aligned} \right\}$. Setting $\lambda = 1$, the first equation gives $w = 0$, and then the third equation gives $u = 0$, while the second equation just gives $v = v$. Thus, we can take $\mathbf{x}_1 = \langle 0, 1, 0\rangle^T$.

Next, setting $\lambda = 2$, the first equation gives $u = 4w$, and the second equation gives $v = -2w$, and so we can take $\mathbf{x}_2 = \langle 4, -2, 1\rangle^T$. Finally, setting $\lambda = -3$, the first and third equations both give $u = -w$, while the second equation gives $w = 2v$, and so we can take $\mathbf{x}_3 = \langle 2, -1, -2\rangle^T$.

10. $A = \begin{pmatrix} 1/6 & 1/4 \\ 2/3 & 0 \end{pmatrix}$ has trace equal to 1/6 and $\det(A) = 0 - 2/12 = -1/6$, and so

the characteristic polynomial for A is given by $\lambda^2 - (1/6)\lambda - 1/6 = (\lambda - 1/2)(\lambda + 1/3)$,

which gives $\lambda_1 = 1/2$ and $\lambda_2 = -1/3$ as the eigenvalues for A.

Setting $\mathbf{x} = \langle u, v\rangle^T$ and $\lambda = 1/2$, the equation $A\mathbf{x} = \lambda\mathbf{x}$ gives $u/6 + v/4 = u/2$, or $2u + 3v = 6u$, giving $3v = 4u$, which means $u/v = 3/4$. Rather than using whole numbers, this time I'll use the fractional form of A to give this eigenvector as $\mathbf{x}_1 = \langle 1/4, 1/3\rangle^T$. Similarly, taking $\lambda = -1/3$, we have $u/6 + v/4 = -u/3$, or $2u + 3v = -4u$, and so $v = -2u$, so I'll take $\mathbf{x}_2 = \langle 1/2, -1\rangle^T$. Thus, using these eigenvectors as the columns of the transformation matrix P, we have

$P = \begin{pmatrix} 1/4 & 1/2 \\ 1/3 & -1 \end{pmatrix}$, with $\det(P) = -1/4 - 1/6 = -5/12$ and $P^{-1} = (12/5)\begin{pmatrix} 1 & 1/2 \\ 1/3 & -1/4 \end{pmatrix}$.

Thus, we have $P^{-1}AP = (12/5)\begin{pmatrix} 1 & 1/2 \\ 1/3 & -1/4 \end{pmatrix}\begin{pmatrix} 1/6 & 1/4 \\ 2/3 & 0 \end{pmatrix}\begin{pmatrix} 1/4 & 1/2 \\ 1/3 & -1 \end{pmatrix}$

$= (12/5)\begin{pmatrix} 1/2 & 1/4 \\ -1/9 & 1/12 \end{pmatrix}\begin{pmatrix} 1/4 & 1/2 \\ 1/3 & -1 \end{pmatrix} = (12/5)\begin{pmatrix} 5/24 & 0 \\ 0 & -5/36 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/3 \end{pmatrix}$,

which shows A is symmetric to its eigenvalue matrix and A is diagonalizable.

\# 12. $A = \begin{pmatrix} 3 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{pmatrix}$ is a symmetric matrix, which means all its eigenvalues must be real numbers. This also means A is not only diagonalizable, it must be orthogonally diagonalizable, i.e. there is an orthogonal matrix P, with $\det(P) = 1$ and $P^{-1} = P^T$, such that P^TAP is a diagonal matrix formed from the eigenvalues of A. (Again, the columns of P use the corresponding eigenvectors.)

We have $|\lambda I - A| = \begin{vmatrix} \lambda - 3 & 2 & -2 \\ 2 & \lambda & 1 \\ -2 & 1 & \lambda \end{vmatrix} = (\lambda - 3)\cdot(\lambda^2 - 1) - 2(2\lambda + 2) + (-2)(2 + 2\lambda)$

$$= (\lambda + 1)\cdot[(\lambda - 3)(\lambda - 1) - 4 - 4] = (\lambda + 1)(\lambda^2 - 4\lambda - 5) = (\lambda + 1)^2(\lambda - 5) .$$

Thus, $\lambda = -1$ is a repeated eigenvalue, with the other being $\lambda = 5$. This does not mean A is nondiagonalizable, indeed we know A is diagonalizable because it is a symmetric matrix.

Taking $\mathbf{x} = \langle u, v, w\rangle^T$ and $\lambda_1 = 5$, we have $2u - v = 5w$ and $-2u - w = 5v$ from $A\mathbf{x} = \lambda\mathbf{x}$. Adding these two equations gives $-(v + w) = 5(v + w)$, which shows $v + w = 0$. Thus, we can take $w = 1$, $v = -1$, and $u = 2$, giving $\mathbf{x}_1 = \langle 2, -1, 1\rangle^T$ as our first eigenvector.

Next, setting $\lambda = -1$, we have the three equations $3u - 2v + 2w = -u$, $-2u - w = -v$, and $2u - v = -w$. All three of these equations gives the same condition, $2u = v - w$, and if we take $w = 0$ this gives $\mathbf{x}_2 = \langle 1, 2, 0\rangle^T$, which is orthogonal to $\mathbf{x}_1$.

Now we need a third eigenvector, which also uses $\lambda = -1$. Many of you chose $\mathbf{x} = \langle -1, 0, 2\rangle^T$, which does solve $A\mathbf{x} = -\mathbf{x}$ and is also orthogonal to $\mathbf{x}_1$, but this choice is not orthogonal to $\mathbf{x}_2$. We can use the cross product to find a vector orthogonal to two given vectors, so if we calculate $\mathbf{x}_3 = \mathbf{x}_2 \times \mathbf{x}_1 = \begin{vmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \end{vmatrix} = \langle 2, -1, -5\rangle^T$ we are lucky enough to find another vector which also solves $A\mathbf{x} = -\mathbf{x}$. Note: the textbook says to use the Gram-Schmidt process, but if the cross product happens to also give a solution then we might as well use it. Thus, using the eigenvectors $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ as the columns of the transformation matrix we have

$P = \begin{pmatrix} 2 & 1 & 2 \\ -1 & 2 & -1 \\ 1 & 0 & -5 \end{pmatrix}$, which does have orthogonal columns but is not an orthogonal matrix because $\det(P) = -30$ and $P^{-1} = (1/30)\begin{pmatrix} 10 & -5 & 5 \\ 6 & 12 & 0 \\ 2 & -1 & -5 \end{pmatrix}$, not the same as P^T.

In order to make P orthogonal we would have to make unit vectors out of $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, which would involve a lot of square roots. However, using P as I have written it we do have $P^{-1}AP$ equal to the diagonal matrix with 5, –1, –1 along the diagonal, as required.

42. $A = \begin{pmatrix} 8 & 15 \\ 15 & -8 \end{pmatrix}$ is another symmetric matrix, with trace equal to zero and

$\det(A) = -64 - 225 = -289 = -(17^2)$, and so the eigenvalues for A are given by

$\lambda_1 = 17$ and $\lambda_2 = -17$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$ and $\mathbf{x}_2 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$.

The eigenvectors are orthogonal, and both have norm equal to the square root of 34. Thus, letting α denote the square root of 34, the transformation matrix is given by

$P = (1/\alpha) \begin{pmatrix} 5 & -3 \\ 3 & 5 \end{pmatrix}$, which is orthogonal with $\det(P) = 1$ and $P^{-1} = P^T$.

We also have $P^TAP = (1/34) \begin{pmatrix} 5 & 3 \\ -3 & 5 \end{pmatrix} \begin{pmatrix} 8 & 15 \\ 15 & -8 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 3 & 5 \end{pmatrix}$

$= (1/34) \begin{pmatrix} 85 & 51 \\ 51 & -85 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 3 & 5 \end{pmatrix} = (1/2) \begin{pmatrix} 5 & 3 \\ 3 & -5 \end{pmatrix} \begin{pmatrix} 5 & -3 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 17 & 0 \\ 0 & -17 \end{pmatrix}$,

as required.

44. Oh, I am so tired from typing all of this up. Most of you who got this far got this problem correct, a 3 x 3 matrix with eigenvalues 0, –3, 6 and corresponding eigenvectors $\langle 1, 0, 1\rangle^T$, $\langle 0, 1, 0\rangle^T$, $\langle 1, 0, -1\rangle^T$, which form an orthogonal basis for $\mathbf{R}^3$. Letting α denote the square root of 2, the orthogonal transformation matrix is given by

$P = (1/\alpha) \begin{pmatrix} 1 & 0 & 1 \\ 0 & \alpha & 0 \\ 1 & 0 & -1 \end{pmatrix}$, which is symmetric and is equal to its own inverse,

and it is easy to verify that $P^{-1}AP$ has the required diagonal form.