University of Memphis
MATH 3242
Linear Algebra
Spring 2025
Dwiggins

Homework Assignment # 3

Chapter Four:
Vector Spaces

4. $\mathbf{u} = \langle 0, 1, -1, 2\rangle$, $\mathbf{v} = \langle 1, 0, 0, 2\rangle$

(a) $\mathbf{u} + \mathbf{v} = \langle 0+1, 1+0, -1+0, 2+2\rangle = \langle 1, 1, -1, 4\rangle$ (b) $2\mathbf{v} = \langle 2, 0, 0, 4\rangle$

(c) $u - \mathbf{v} = \langle -1, 1, -1, 0\rangle$ (d) $3\mathbf{u} - 2\mathbf{v} = \langle 0, 3, -3, 6\rangle - \langle 2, 0, 0, 4\rangle = \langle -2, 3, -3, 2\rangle$

8. $\mathbf{u} = \langle 1, -1, 2\rangle$, $\mathbf{v} = \langle 0, 2, 3\rangle$, $\mathbf{w} = \langle 0, 1, 1\rangle$

If $3\mathbf{u} + 2\mathbf{x} = \mathbf{w} - \mathbf{v}$, then $\langle 3, -3, 6\rangle + 2\mathbf{x} = \langle 0, -1, -2\rangle$,

which gives $2\mathbf{x} = \langle -3, 2, -8\rangle$, and so $\mathbf{x} = \langle -1.5, 1, -4\rangle$.

12. $\mathbf{u}_1 = \langle 1, -2, 1, 1\rangle$, $\mathbf{u}_2 = \langle -1, 2, 3, 2\rangle$, $\mathbf{u}_3 = \langle 0, -1, -1, -1\rangle$

We can use inner products to show $\mathbf{u}_1$ is orthogonal to both $\mathbf{u}_2$ and $\mathbf{u}_3$:

$\mathbf{u}_1 \bullet \mathbf{u}_2 = -1 - 4 + 3 + 2 = 0$ and $\mathbf{u}_1 \bullet \mathbf{u}_3 = 0 + 2 - 1 - 1 = 0$, while $\mathbf{u}_2 \bullet \mathbf{u}_3 = 0 - 2 - 3 - 2 = -7$.

Also note that if we form a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, the way to get 0 as the first component is to simply add $\mathbf{u}_1 + \mathbf{u}_2$, and this does not give $\mathbf{u}_3$. Thus, these three vectors are linearly independent, and they span a three-dimensional subspace of $\mathbf{R}^4$. Finally, we want the three values of $\mathbf{u}_1 \bullet \mathbf{u}_1 = 1 + 4 + 1 + 1 = 7$, $\mathbf{u}_2 \bullet \mathbf{u}_2 = 1 + 4 + 9 + 4 = 18$, and $\mathbf{u}_3 \bullet \mathbf{u}_3 = 0 + 1 + 1 + 1 = 3$.

Now let $\mathbf{v} = \langle 4, -13, -5, -4\rangle$, and calculate $\mathbf{v} \bullet \mathbf{u}_1 = 21$, $\mathbf{v} \bullet \mathbf{u}_2 = -53$, $\mathbf{v} \bullet \mathbf{u}_3 = 22$.

We want to see if $\mathbf{v}$ lies in the space spanned by $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, which means we want to find scalars c_1, c_2, c_3 which will give $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$. Because $\mathbf{u}_1$ is orthogonal to both $\mathbf{u}_2$ and $\mathbf{u}_3$, the easiest way to find c_1 is to take the inner product of both sides of this equation with $\mathbf{u}_1$, which gives $\mathbf{v} \bullet \mathbf{u}_1 = c_1\mathbf{u}_1 \bullet \mathbf{u}_1 + 0 + 0$, or $21 = 7c_1$, and so we have $c_1 = 3$.

Next, we have $\mathbf{v} \bullet \mathbf{u}_2 = 0 + c_2\mathbf{u}_2 \bullet \mathbf{u}_2 + c_3\mathbf{u}_3 \bullet \mathbf{u}_2$, which gives $18c_2 - 7c_3 = -53$, and $\mathbf{v} \bullet \mathbf{u}_3 = 0 + c_2\mathbf{u}_2 \bullet \mathbf{u}_3 + c_3\mathbf{u}_3 \bullet \mathbf{u}_3$, which gives $-7c_2 + 3c_3 = 22$. Solving this 2 x 2 system of equations gives $c_2 = -1$ and $c_3 = 5$, and so we have $\mathbf{v} = 3\mathbf{u}_1 - \mathbf{u}_2 + 5\mathbf{u}_3$, which does check out correctly.

38. $A = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}$ implies $\det(A) = 2 - 12 = -10$ is not equal to zero,

which means A is a matrix of full rank. Thus, the null space of A is simply $\{0\}$ (since the only solution to $AX = 0$ is $X = 0$), the nullity of A is zero, and the rank of A is 2.

40. $A = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 4 & -2 & 4 & -2 \\ -2 & 0 & 1 & 3 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & -2 & 12 & -2 \\ 0 & 0 & -3 & 3 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -6 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

If we set $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ and solve the homogeneous equation $AX = 0$, then the bottom row in the reduced row echelon matrix shows we must have $x_4 = x_3$, while the top row gives $x_1 = 2x_3$. Substituting these into the middle row gives $x_2 = 5x_3$, and so the null space for A is the one-dimensional subspace of $\mathbf{R}^4$ spanned by $\begin{pmatrix} 2 \\ 5 \\ 1 \\ 1 \end{pmatrix}$.

We can also see from the reduced row echelon matrix that the row space of A is spanned by three vectors in $\mathbf{R}^4$, and so we have the nullity and rank of A satisfying $1 + 3 = 4$, which gives the number of columns of A, as required.

44. $A = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 5 & 6 \\ 1 & 16 & 14 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 2 & -1 & 4 \\ 0 & 5.5 & 4 \\ 0 & 16.5 & 12 \end{pmatrix}$ This row reduction is obtained using –0.5*R1 + R2 = New R2 and –0.5*R1 + R3 = New R3.

In the reduced row echelon matrix, the third row is three times the second row, and so the rank of A is equal to 2. The row space is spanned by the first two rows of A.

58. The linearly independent vectors in $B' = \{\langle 2, 2\rangle, \langle 0, -1\rangle\}$ form a two-dimensional subspace of $\mathbf{R}^2$. The standard unit vectors in $\mathbf{R}^2$ are $\langle 1,0\rangle$ and $\langle 0,1\rangle$. In terms of B', we have $\langle 1,0\rangle = 0.5*\langle 2, 2\rangle + 1*\langle 0, -1\rangle$ and $\langle 0,1\rangle = 0*\langle 2, 2\rangle + (-1)*\langle 0,-1\rangle$.

Thus, $\mathbf{x} = \langle -1, 2\rangle = (-1)*\langle 1,0\rangle + 2*\langle 0,1\rangle = (-0.5 + 0)*\langle 2, 2\rangle + (-1 - 2)*\langle 0, -1\rangle$,

and so the coordinates of $\mathbf{x}$ relative to the given basis are given by $[\mathbf{x}]_{B'} = \begin{pmatrix} -0.5 \\ -3 \end{pmatrix}$.